A.1. Proofs of Main Results

This Appendix presents the proofs of some of the results presented in the previous sections.

A.1.1. Proof of Lemma 2.1 We first show the claim for $\alpha < \frac{1}{2}$, that is,

$$ (1 - \alpha) I_\alpha (P, Q) - \frac{1}{2} I_\frac{1}{2} (P, Q) \geq 0. \quad (A.1) $$

Let $H_\alpha (x) = \frac{1}{\alpha} (1 - x^\alpha) - 2 \left(1 - x^{\frac{1}{2}} \right), \ 0 \leq x \leq \infty$, then the above inequality becomes

$$ \int H_\alpha \left( \frac{p}{q} \right) q \, d\nu \geq 0. \quad (A.2) $$

Note

$$ \frac{d}{dx} H_\alpha (x) = -x^{\alpha-1} + x^{-\frac{1}{2}} \begin{cases} > 0 \quad \text{if } x > 1 \\ = 0 \quad \text{if } x = 1 \\ < 0 \quad \text{if } x < 1. \end{cases} $$

The above holds for the case with $\alpha = 0$ as well, since $H_0 (x) = -\log x - 2 \left(1 - x^{\frac{1}{2}} \right)$. Moreover, $H_\alpha (1) = 0$. Therefore $H_\alpha (x) \geq 0$ for all $x \geq 0$, and the desired inequality (A.2) follows immediately. Next, we prove the case with $\alpha > \frac{1}{2}$, that is,

$$ \alpha I_\alpha (P, Q) \geq \frac{1}{2} I_\frac{1}{2} (P, Q). $$

Let $\beta = 1 - \alpha < \frac{1}{2}$, then the above inequality becomes

$$ (1 - \beta) I_{1-\beta} (P, Q) \geq \frac{1}{2} I_\frac{1}{2} (P, Q). \quad (A.3) $$

By (A.1) and the symmetry of the Hellinger distance,

$$ (1 - \beta) I_{\beta} (Q, P) \geq \frac{1}{2} I_\frac{1}{2} (Q, P) = \frac{1}{2} I_\frac{1}{2} (P, Q). $$

But the equality $I_{1-\beta} (P, Q) = I_\beta (Q, P)$ holds for every $\beta \in \mathbb{R}$, and (A.3) follows.
Notation. Let $C$ be a generic positive constant, $\| \cdot \|$ be the $L_2$-metric,

\[
\theta_n = \theta_0 + t/\sqrt{n}, \quad \bar{T}_{Q_n} = \bar{T}(Q_n), \quad \bar{T}_{P_n} = \bar{T}(P_n),
\]

\[
\bar{P}_{\theta_n} = \arg \min_{P \in \mathcal{P}_n} H(P, Q), \quad R_n(Q, \theta, \gamma) = -\int \frac{1}{(1 + \gamma' g_n(x, \theta))} dQ,
\]

\[
g_n(x, \theta) = g(x, \theta) \mathbb{1}_{\{x \in \mathcal{X}_n\}}, \quad \Lambda_n = G'\Omega^{-1} g_n(x, \theta_0), \quad \Lambda = G'\Omega^{-1} g(x, \theta_0),
\]

\[
\psi_{n, Q_n} = -2 \left( \int \Lambda_n \Lambda_n' dQ_n \right)^{-1} \int \Lambda_n \left\{ dQ_n^{1/2} - d\bar{P}_{\theta_n}^{1/2} \right\} dQ_n^{1/2}.
\]

A.1.2. Proof of Theorem 3.1

A.1.2.1. Proof of (i). Pick arbitrary $r > 0$ and $t \in \mathbb{R}^p$. Consider the following parametric submodel having the likelihood ratio

\[
(A.4) \quad \frac{dP_{\theta_n, \zeta_n}}{dP_0} = \frac{1 + \zeta_n' g_n(x, \theta_n)}{\int (1 + \zeta_n' g_n(x, \theta_n)) dP_0} = f(x, \theta_n, \zeta_n),
\]

where

\[
\zeta_n = -E_{P_0} \left[ g(x, \theta_n) g_n(x, \theta_n) \right]' E_{P_0} \left[ g(x, \theta_n) \right] .
\]

Note that $P_{\theta_0, 0} = P_0, P_{\theta_n, \zeta_n} \in \mathcal{P}_{\theta_n}$ (by the definition of $\zeta_n$), and $\zeta_n = O(n^{-1/2})$ (by the proof of Lemma A.4(i)). Also, since $\sup_{x \in \mathcal{X}} |\zeta_n' g_n(x, \theta_n)| = O(n^{-1/2}m_n) = o(1)$, the likelihood ratio $\frac{dP_{\theta_n, \zeta_n}}{dP_0}$ is well-defined for all $n$ large enough. So, for this submodel the mapping $T_a$ must satisfy (3.1).

We now evaluate the Hellinger distance between $P_{\theta_n, \zeta_n}$ and $P_0$. An expansion around $\zeta_n = 0$ yields

\[
H(P_{\theta_n, \zeta_n}, P_0) = \left\| \frac{\zeta_n}{\partial f(x, \theta_n, \zeta_n)^{1/2}} \right\|_{\zeta_n = 0}^{dP_{\theta_n, \zeta_n}/dP_0} + \frac{1}{2} \left\| \frac{\zeta_n}{\partial f(x, \theta_n, \zeta_n)^{1/2}} \right\|_{\zeta_n = \hat{\zeta}_n}^{dP_{\theta_n, \zeta_n}/dP_0},
\]

where $\hat{\zeta}_n$ is a point on the line joining $\zeta_n$ and 0, and

\[
\frac{\partial f(x, \theta_n, \zeta_n)^{1/2}}{\partial \zeta_n} \bigg|_{\zeta_n = 0} = \frac{1}{2} \{ g_n(x, \theta_n) - E_{P_0} [g_n(x, \theta_n)] \} ,
\]

\[
\frac{\partial^2 f(x, \theta_n, \zeta_n)^{1/2}}{\partial \zeta_n \partial \zeta_n} \bigg|_{\zeta_n = 0} = -\frac{1}{4} (1 + \zeta_n' g_n(x, \theta_n))^{-3/2} (1 + \zeta_n' E_{P_0} [g_n(x, \theta_n)])^{-1/2} g_n(x, \theta_n) g_n(x, \theta_n)'
\]

\[-\frac{1}{2} (1 + \zeta_n' g_n(x, \theta_n))^{-1/2} (1 + \zeta_n' E_{P_0} [g_n(x, \theta_n)])^{-3/2} g_n(x, \theta_n) E_{P_0} [g_n(x, \theta_n)']
\]

\[+ \frac{3}{4} (1 + \zeta_n' g_n(x, \theta_n))^{1/2} (1 + \zeta_n' E_{P_0} [g_n(x, \theta_n)])^{-5/2} E_{P_0} [g_n(x, \theta_n)] E_{P_0} [g_n(x, \theta_n)'] .
\]
Thus, a lengthy but straightforward calculation combined with Lemma \[\text{A.3}\] \(\zeta_n = O \left( n^{-1/2} \right)\), and \(\sup_{x \in X} |\zeta'_n g_n(x, \theta_n)| = o(1)\) implies

\[
(A.5) \quad nH(P_{\theta_n, \zeta_n}, P_0)^2 = n \left\| \frac{1}{2} \zeta'_n (g_n(x, \theta_n) - EP_0 [g_n(x, \theta_n)]) dP_0^{1/2} \right\|^2 + o(1) \rightarrow \frac{1}{4} t'\Sigma^{-1} t.
\]

Based on this limit, a lower bound of the maximum bias of \(T_a\) is obtained as (see, Rieder (1994, eq. (56) on p. 180))

\[
\liminf_{n \to \infty} \sup_{Q \in B_H(P_0, r/\sqrt{n})} n (\tau \circ T_a(Q) - \tau(\theta_0))^2 \\
\geq \liminf_{n \to \infty} \sup_{\{t \in \mathbb{R}^p : P_{\theta_n, \zeta_n} \in B_H(P_0, r/\sqrt{n})\}} n (\tau \circ T_a(P_{\theta_n, \zeta_n}) - \tau(\theta_0))^2 \\
\geq \max_{\{t \in \mathbb{R}^p : \frac{1}{4} t' \Sigma t \leq r^2 - \epsilon\}} \left( \left( \frac{\partial \tau(\theta_0)}{\partial \theta} \right)' \right)^2 = 4(r^2 - \epsilon) \left( \frac{\partial \tau(\theta_0)}{\partial \theta} \right)' \Sigma^{-1} \left( \frac{\partial \tau(\theta_0)}{\partial \theta} \right),
\]

for each \(\epsilon \in (0, r^2)\), where the first inequality follows from the set inclusion relationship, the second inequality follows from \(\text{A.1}\) and \(\text{A.5}\), and the equality follows from the Kuhn-Tucker theorem. Since \(\epsilon\) can be arbitrarily small, we obtain the conclusion.

A.1.2.2. Proof of (ii). Pick arbitrary \(r > 0\) and sequence \(Q_n \in B_H(P_0, r/\sqrt{n})\). We first show the Fisher consistency of \(\hat{T}\). From Lemma \[\text{A.2}\] (note: \(P_{\theta_n, \zeta_n} \in B_H(P_0, r/\sqrt{n})\) for all \(n\) large enough),

\[
\sqrt{n} \left( \hat{T} (P_{\theta_n, \zeta_n}) - \theta_0 \right) = -\sqrt{n} \Sigma^{-1} \int \Lambda_n dP_{\theta_n, \zeta_n} + o(1)
\]

\[
= \Sigma^{-1} G' \Omega^{-1} \int \partial g(x, \hat{\theta}) / \partial \theta dP_{\theta_n, \zeta_n} t + o(1)
\]

\[
\rightarrow t
\]

for all \(n\) large enough, where \(\hat{\theta}\) is a point on the line joining \(\theta_n\) and \(\theta_0\), the second equality follows from \(\int g(x, \theta_0) \mathbb{I} \{x \notin X_n\} dP_{\theta_n, \zeta_n} = o \left( n^{-1/2} \right)\) (by a similar argument to \(\text{A.1}\)), \(\int g(x, \theta_n) dP_{\theta_n, \zeta_n} = 0\) (by \(P_{\theta_n, \zeta_n} \in P_{\theta_a}\)), and an expansion around \(\theta_n = \theta_0\), and the convergence follows from the last statement of Lemma \[\text{A.4}\] (i). Therefore, \(\hat{T}\) is Fisher consistent.

We next show \(\text{A.1}\). An expansion of \(\tau \circ \hat{T}_{Q_n}\) around \(\hat{T}_{Q_n} = \theta_0\), Lemmas \[\text{A.1}\] (ii) and \[\text{A.2}\] and Assumption \[\text{A.1}\] (viii) imply

\[
\sqrt{n} \left( \tau \circ \hat{T}_{Q_n} - \tau(\theta_0) \right) = -\sqrt{n} \left( \frac{\partial \tau(\theta_0)}{\partial \theta} \right)' \Sigma^{-1} \int \Lambda_n dQ_n + o(1)
\]

\[
= -\sqrt{n} \nu' \int \Lambda_n \left\{ dQ_n^{1/2} - dP_0^{1/2} \right\} dQ_n^{1/2} - \sqrt{n} \nu' \int \Lambda_n dP_0^{1/2} \left\{ dQ_n^{1/2} - dP_0^{1/2} \right\} + o(1),
\]
where we denote \( \nu' = \left( \frac{\partial \tau(\theta_0)}{\partial \theta} \right)' \). From the triangle inequality,

\[
\begin{align*}
    n \left( \tau \circ \bar{T}_{Q_n} - \tau(\theta_0) \right)^2 \\
    \leq n \left\{ \left| \nu' \int \Lambda_n \left\{ dQ_n^{1/2} - dP_0^{1/2} \right\} dQ_n \right|^2 + \left| \nu' \int \Lambda_n \left\{ dQ_n^{1/2} - dP_0^{1/2} \right\} dP_0^{1/2} \right|^2 \right\} + o(1) \\
    = n \{ A_1 + A_2 + 2A_3 \} + o(1). 
\end{align*}
\]

For \( A_1 \), observe that

\[
A_1 \leq \left| \nu' \int \Lambda_n \Lambda'_n dQ_n \nu_0 \right| \left( \int \left\{ dQ_n^{1/2} - dP_0^{1/2} \right\} \right)^2 \leq B^* \frac{r^2}{n} + o(n^{-1}),
\]

where the first inequality follows from the Cauchy-Schwarz inequality, and the second inequality follows from Lemma \( A.5 \) (i) and \( Q_n \in B_H(P_0, r/\sqrt{n}) \). Similarly, we have \( A_2 \leq B^* \frac{r^2}{n} + o(n^{-1}) \) and \( A_3 \leq B^* \frac{r^2}{n} + o(n^{-1}) \). Combining these terms,

\[
(A.6) \quad \limsup_{n \to \infty} \left( \tau \circ \bar{T}_{Q_n} - \tau(\theta_0) \right)^2 \leq 4r^2 B^*,
\]

for any sequence \( Q_n \in B_H(P_0, r/\sqrt{n}) \) and \( r > 0 \). Pick any \( r > 0 \). Since the supremum

\[
\sup_{Q \in B_H\left( P_0, \frac{r}{\sqrt{n}} \right)} n \left( \tau \circ \bar{T}(Q) - \tau(\theta_0) \right)^2
\]

is finite for all \( n \) large enough (from Lemma \( A.1 \) (i)), there exists a sequence \( \bar{Q}_n \in B_H\left( P_0, \frac{r}{\sqrt{n}} \right) \) such that

\[
\limsup_{n \to \infty} \left( \tau \circ \bar{T}_{Q_n} - \tau(\theta_0) \right)^2 = \limsup_{n \to \infty} \sup_{Q \in B_H\left( P_0, \frac{r}{\sqrt{n}} \right)} n \left( \tau \circ \bar{T}(Q) - \tau(\theta_0) \right)^2.
\]

Therefore, the conclusion follows by \( (A.6) \).

**A.1.3. Proof of Theorem 3.2.**

**A.1.3.1. Proof of (i).** Pick arbitrary \( \epsilon \in (0, r^2) \) and \( r > 0 \). Consider the parametric submodel \( P_{\theta_n, \zeta_n} \) defined in \( (A.4) \). The convolution theorem (Theorem 25.20 of van der Vaart (1998)) implies that for each \( t \in \mathbb{R}^p \), there exists a probability measure \( M_0 \) which does not depend on \( t \) and satisfies

\[
(A.7) \quad \sqrt{n} \left( \tau \circ T_a(P_n) - \tau \circ T_a(P_{\theta_n, \zeta_n}) \right) \overset{d}{\to} M_0 \ast N(0, B^*) \quad \text{under } P_{\theta_n, \zeta_n}.
\]

Let

\[
t^* = \arg \max_{\{t \in \mathbb{R}^p : \frac{1}{2} t' \Sigma t \leq r^2 - \epsilon \}} \left( \frac{\partial \tau(\theta_0)}{\partial \theta} \right)' t^2 \quad \text{s.t.} \quad \left( \frac{\partial \tau(\theta_0)}{\partial \theta} \right)' \int \xi dM_0 \ast N(0, B^*) \geq 0.
\]

Since the integral \( \int \xi dM_0 \ast N(0, B^*) \) does not depend on \( t \), such \( t^* \) always exists. From \( \frac{1}{4} t^* \Sigma t^* \leq r^2 - \epsilon \) and \( (A.5) \), it holds that \( P_{\theta_0 + t^*/\sqrt{n}, \zeta_n} \in B_H(P_0, r/\sqrt{n}) \) for all \( n \) large enough. Also, note that
Thus, $P_{\theta_0 + t^*/\sqrt{m_0}} \in \Bar{B}_H (P_0, r/\sqrt{n})$ for all $n$ large enough, and we have
\[
\lim_{b \to \infty} \lim_{n \to \infty} \sup_{Q \in B_H (P_0, r/\sqrt{n})} \int b \wedge n (\tau \circ T_a (P_n) - \tau (\theta_0))^2 dQ^\otimes n \geq \lim_{b \to \infty} \lim_{n \to \infty} \int b \wedge n (\tau \circ T_a (P_n) - \tau (\theta_0))^2 dP_{\theta_0 + t^*/\sqrt{m_0}}^\otimes \]
\[
= \lim_{b \to \infty} \lim_{n \to \infty} \int b \wedge n \left( \xi + \left( \frac{\partial \tau (\theta_0)}{\partial \theta} \right) t^* \right)^2 dM_0 \ast N (0, B^*) \]
\[
= \int \xi^2 dM_0 \ast N (0, B^*) + \left( \left( \frac{\partial \tau (\theta_0)}{\partial \theta} \right) t^* \right)^2 + 2 \left( \frac{\partial \tau (\theta_0)}{\partial \theta} \right) t^* \int \xi dM_0 \ast N (0, B^*) \geq \{1 + 4 (r^2 - \epsilon)\} B^*,
\]
where the first equality follows from the Fisher consistency of $T_a$, (A.9), and the continuous mapping theorem, the second equality follows from the monotone convergence theorem, and the second inequality follows from the definition of $t^*$. Since $\epsilon$ can be arbitrarily small, we obtain the conclusion.

A.1.3.2. Proof of (ii). Pick arbitrary $r > 0$ and $b > 0$. Applying the inequality $b \wedge (c_1 + c_2) \leq b \wedge c_1 + b \wedge c_2$ for any $c_1, c_2 \geq 0$,
\[
\limsup_{n \to \infty} \sup_{Q \in B_H (P_0, r/\sqrt{n})} \int b \wedge n (\tau \circ T (P_n) - \tau (\theta_0))^2 dQ^\otimes n \leq \limsup_{n \to \infty} \sup_{Q \in B_H (P_0, r/\sqrt{n})} \int b \wedge \{ n |\tau \circ T (P_n) - \tau \circ \Bar{T} (P_n) | \tau \circ T (P_n) - \tau (\theta_0) | \} dQ^\otimes n \]
\[
+ 2 \limsup_{n \to \infty} \sup_{Q \in B_H (P_0, r/\sqrt{n})} \int b \wedge n (\tau \circ \Bar{T} (P_n) - \tau (\theta_0))^2 dQ^\otimes n
\]
(A.8) $= A_1 + 2A_2 + A_3$.

For $A_1$,
\[
A_1 \leq b \times \limsup_{n \to \infty} \sup_{Q \in B_H (P_0, r/\sqrt{n})} \int_{(x_1, \ldots, x_n) \notin X_n^m} dQ^\otimes n \leq b \times \limsup_{n \to \infty} \sup_{Q \in B_H (P_0, r/\sqrt{n})} \sum_{i=1}^n \int_{x_i \notin X_n} dQ
\]
(A.9) $\leq b \times \limsup_{n \to \infty} \sup_{Q \in B_H (P_0, r/\sqrt{n})} nm_n^{-\eta} E_Q \left[ \sup_{\theta \in \Theta} g (x, \theta)^\eta \right] = 0,$
where the first inequality follows from \( T(P_n) = \bar{T}(P_n) \) for all \((x_1, \ldots, x_n) \in \mathcal{X}_n^{\omega}\), the second inequality follows from a set inclusion relation, the third inequality follows from the Markov inequality, and the equality follows from Assumption [3.1] (vii) and \( E_Q [\sup_{\theta \in \Theta} |g(x, \theta)| \eta] < \infty \) for all \( Q \in \bar{B}_H(P_0, r/\sqrt{n}) \). Similarly, we have \( A_2 = 0 \).

We now consider \( A_3 \). Note that the mapping \( f_{b,n}(Q) = \int b \wedge n (\tau \circ \bar{T}(P_n) - \tau(\theta_0))^2 dQ_{\otimes n} \) is continuous in \( Q \in B_H(P_0, r/\sqrt{n}) \) under the Hellinger distance for each \( n \), and the set \( B_H(P_0, r/\sqrt{n}) \) (not \( \bar{B}_H(P_0, r/\sqrt{n}) \)) is compact under the Hellinger distance for each \( n \). Thus, there exists \( \tilde{Q}_{b,n} \in B_H(P_0, r/\sqrt{n}) \) such that \( \sup_{Q \in B_H(P_0, r/\sqrt{n})} f_n(Q) = f_n(\tilde{Q}_{b,n}) \) for each \( n \). Then we have

\[
A_3 \leq \limsup_{n \to \infty} \sup_{Q \in B_H(P_0, r/\sqrt{n})} \int b \wedge n (\tau \circ \bar{T}(P_n) - \tau(\theta_0))^2 dQ_{\otimes n}
= \limsup_{n \to \infty} \int b \wedge n (\tau \circ \bar{T}(P_n) - \tau(\theta_0))^2 d\tilde{Q}_{b,n}^{\otimes n}
= \int b \wedge (\xi + \tilde{t}_b)^2 dN(0, B^*)
\leq B^* + \tilde{t}_b^2
\leq (1 + 4\tilde{t}_b^2) B^*,
\]

where \( \tilde{t}_b = \limsup_{n \to \infty} \sqrt{n} \left( \tau \circ \bar{T} \left( \tilde{Q}_{b,n} \right) - \tau(\theta_0) \right) \), the first inequality follows from \( \bar{B}_H(P_0, r/\sqrt{n}) \subseteq B_H(P_0, r/\sqrt{n}) \), the second equality follows from Lemma A.8 (with \( Q_n = \tilde{Q}_{b,n} \)) and the continuous mapping theorem, the second inequality follows from \( b \wedge c \leq c \) and a direct calculation, and the last inequality follows from Theorem 3.1(ii). Combining these results, the conclusion is obtained.

A.1.4. Proof of Theorem 3.3

A.1.4.1. Proof of (i). Consider the parametric submodel \( P_{\theta_n, \zeta_n} \) defined in [A.4]. Since \( \ell \) is uniformly continuous on \( \mathbb{R}^p \) (by Assumption 3.2) and \( T_a \) is Fisher consistent,

\[
b \wedge \ell \left( \sqrt{n} \{ S_n - \tau \circ T_a(P_{\theta_n, \zeta_n}) \} \right) - b \wedge \ell \left( \sqrt{n} \{ S_n - \tau(\theta_0) \} - \left( \frac{\partial \tau(\theta_0)}{\partial \theta} \right)' \right) \to 0,
\]

uniformly in \( t, |t| < c \) and \( \{S_n\}_{n \in \mathbb{N}} \) for each \( c > 0 \) and \( b > 0 \). Thus,

\[
\inf_{S_n \in S} \sup_{|t| \leq c} \int b \wedge \ell \left( \sqrt{n} \{ S_n - \tau \circ T_a(P_{\theta_n, \zeta_n}) \} \right) dP_{\theta_n, \zeta_n}^{\otimes n} = \inf_{R_n \in R} \sup_{|t| \leq c} \int b \wedge \ell \left( R_n - \left( \frac{\partial \tau(\theta_0)}{\partial \theta} \right)' \right) dP_{\theta_n, \zeta_n}^{\otimes n} + o(1),
\]

where the first inequality follows from \( T(P_n) = \bar{T}(P_n) \) for all \((x_1, \ldots, x_n) \in \mathcal{X}_n^{\omega}\), the second inequality follows from a set inclusion relation, the third inequality follows from the Markov inequality, and the equality follows from Assumption [3.1] (vii) and \( E_Q [\sup_{\theta \in \Theta} |g(x, \theta)| \eta] < \infty \) for all \( Q \in \bar{B}_H(P_0, r/\sqrt{n}) \). Similarly, we have \( A_2 = 0 \).

We now consider \( A_3 \). Note that the mapping \( f_{b,n}(Q) = \int b \wedge n (\tau \circ \bar{T}(P_n) - \tau(\theta_0))^2 dQ_{\otimes n} \) is continuous in \( Q \in B_H(P_0, r/\sqrt{n}) \) under the Hellinger distance for each \( n \), and the set \( B_H(P_0, r/\sqrt{n}) \) (not \( \bar{B}_H(P_0, r/\sqrt{n}) \)) is compact under the Hellinger distance for each \( n \). Thus, there exists \( \tilde{Q}_{b,n} \in B_H(P_0, r/\sqrt{n}) \) such that \( \sup_{Q \in B_H(P_0, r/\sqrt{n})} f_n(Q) = f_n(\tilde{Q}_{b,n}) \) for each \( n \). Then we have

\[
A_3 \leq \limsup_{n \to \infty} \sup_{Q \in B_H(P_0, r/\sqrt{n})} \int b \wedge n (\tau \circ \bar{T}(P_n) - \tau(\theta_0))^2 dQ_{\otimes n}
= \limsup_{n \to \infty} \int b \wedge n (\tau \circ \bar{T}(P_n) - \tau(\theta_0))^2 d\tilde{Q}_{b,n}^{\otimes n}
= \int b \wedge (\xi + \tilde{t}_b)^2 dN(0, B^*)
\leq B^* + \tilde{t}_b^2
\leq (1 + 4\tilde{t}_b^2) B^*,
\]

where \( \tilde{t}_b = \limsup_{n \to \infty} \sqrt{n} \left( \tau \circ \bar{T} \left( \tilde{Q}_{b,n} \right) - \tau(\theta_0) \right) \), the first inequality follows from \( \bar{B}_H(P_0, r/\sqrt{n}) \subseteq B_H(P_0, r/\sqrt{n}) \), the second equality follows from Lemma A.8 (with \( Q_n = \tilde{Q}_{b,n} \)) and the continuous mapping theorem, the second inequality follows from \( b \wedge c \leq c \) and a direct calculation, and the last inequality follows from Theorem 3.1(ii). Combining these results, the conclusion is obtained.
for each $c > 0$, where $R_n = \sqrt{n} \{ S_n - \tau(\theta_0) \}$ is a standardized estimator and $R = \{ \sqrt{n} \{ S_n - \tau(\theta_0) \} : S_n \in S \}$. By expanding the log likelihood ratio $\log \frac{dP_{\theta_n, \hat{c}_n}}{dP_0^{\otimes n}}$ around $\hat{c}_n = 0$,

$$
\log \frac{dP_{\theta_n, \hat{c}_n}}{dP_0^{\otimes n}} = \hat{c}_n' \sum_{i=1}^{n} \{ g_n(x_i, \theta_n) - E_{P_0}[g_n(x, \theta_n)] \} - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{ g_n(x_i, \theta_n) g_n(x_i, \theta_n) \} \frac{\hat{c}_n}{2} + \frac{n\hat{c}_n E_{P_0}[g_n(x, \theta_n)] E_{P_0}[g_n(x, \theta_n)]'} \frac{\hat{c}_n}{2} \left( 1 + \hat{c}_n g_n(x, \theta_n) \right)^2
$$

$$
= L_1 - L_2 + L_3.
$$

where $\hat{c}_n$ and $\tilde{c}_n$ are points on the line joining $\hat{c}_n$ and 0. For $L_1$, an expansion of $g_n(x, \theta_n)$ (in $\hat{c}_n$) around $\theta_n = \theta_0$ combined with Lemma A.4 (i) implies that under $P_0$,

$$
L_1 = -t' G\Omega^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{ g_n(x_i, \theta_n) - E_{P_0}[g_n(x, \theta_n)] \} + o_p(1).
$$

Also, Lemma A.4 (i) and $\sup_{x \in \mathcal{X}} |\hat{c}_n g_n(x, \theta_n)| = o(1)$ imply that under $P_0$,

$$
L_2 \overset{P}{\to} \frac{1}{2} t' \Sigma t, \quad L_3 \to 0.
$$

Therefore, in the terminology of Rieder (1994, Definition 2.2.9), the parametric model $P_{\theta_n, \hat{c}_n}$ is asymptotically normal with the asymptotic sufficient statistic $-G\Omega^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{ g_n(x_i, \theta_n) - E_{P_0}[g_n(x, \theta_n)] \}$ and the asymptotic covariance matrix $\Sigma$. Note that this is essentially the LAN (local asymptotic normality) condition introduced by LeCam. If $P_{\theta_n, \hat{c}_n}$ is asymptotically normal in this sense, we can directly apply the result of the minimax risk bound by Rieder (1994, Theorem 3.3.8 (a)), that is

$$
(A.11) \lim_{b \to \infty} \lim_{c \to \infty} \inf_{n \to \infty} \sup_{S \in \mathcal{S}} \int b \wedge \ell \left( \sqrt{n} \{ S_n - \tau(\theta_0) \} \right) dP_{\theta_n, \hat{c}_n}^{\otimes n} \geq \int \ell dN(0, B^*)
$$

(see also Theorem 1 in LeCam and Yang (1990)). From (A.10) and (A.11),

$$
\lim_{b \to \infty} \lim_{c \to \infty} \inf_{n \to \infty} \sup_{S \in \mathcal{S}} \int b \wedge \ell \left( \sqrt{n} \{ S_n - \tau(\theta_0) \} \right) dP_{\theta_n, \hat{c}_n}^{\otimes n} \geq \int \ell dN(0, B^*).
$$

Finally, since $E_{P_{\theta_n, \hat{c}_n}}[\sup_{\theta \in \Theta} |g(x, \theta)|] < \infty$ for all $n$ large enough (by $\sup_{x \in \mathcal{X}} |\hat{c}_n g_n(x, \theta_n)| = o(1)$ and Assumption 3.1 (v)), we have $P_{\theta_n, \hat{c}_n} \in \mathcal{B}_H (P_0, t/\sqrt{n})$ for all $t$ satisfying $\frac{1}{4} t' \Sigma t \leq t^2 - \epsilon$ with any $\epsilon \in (0, t^2)$ and all $n$ large enough. Therefore, the set inclusion relation yields

$$
\lim_{b \to \infty} \lim_{c \to \infty} \inf_{n \to \infty} \sup_{S \in \mathcal{S}} \int b \wedge \ell \left( \sqrt{n} \{ S_n - \tau(\theta_0) \} \right) dQ^{\otimes n} \geq \lim_{b \to \infty} \lim_{c \to \infty} \inf_{n \to \infty} \sup_{S \in \mathcal{S}} \int b \wedge \ell \left( \sqrt{n} \{ S_n - \tau(\theta_0) \} \right) dP_{\theta_n, \hat{c}_n}^{\otimes n},
$$

which implies the conclusion.
In the first step, we show there exists a \( g \) which contradicts with Assumption 3.1 (vi). An argument similar to (A.9) implies that the first term of (A.12) is zero. From (A.12) where the equality follows from Lemma A.8, the uniform continuity of \( B \) in the interior of the convex hull of \( \bar{\theta} \) under Corollary 2.3.1), we can take Lemma A.1.

**Proof of (ii).**

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\[ (\bar{\theta},...,\bar{x}) : \text{for each } \bar{x} \in \mathcal{X}_n \]

The proof is split into several steps. Let \( \mathcal{G}(\theta,Q) \) be the convex hull of the support of \( g(x,\theta) \) under \( x \sim Q \).

In the first step, we show \( \mathcal{G}(\theta,Q) \) exists for all \( Q \in B_H(P_0, r/\sqrt{n}) \) and all \( n \) large enough, \( \bar{T}_Q \to \theta_0 \) as \( n \to \infty \) for each \( r > 0 \) and sequence \( Q_n \in B_H(P_0, r/\sqrt{n}) \).

**Proof of (i).** The proof is split into several steps. Let \( \mathcal{G}(\theta,Q) \) be the convex hull of the support of \( g(x,\theta) \) under \( x \sim Q \).

In the first step, we show \( 0 \in \text{int} \mathcal{G}(\theta_0, P_0) \). If \( 0 \notin \mathcal{G}(\theta_0, P_0) \), then we have \( E_{P_0}[g(x,\theta_0)] \neq 0 \), which is a contradiction. Thus, it is enough to show that 0 is not on the boundary of \( \mathcal{G}(\theta_0, P_0) \). Suppose 0 is indeed on the boundary of \( \mathcal{G}(\theta_0, P_0) \). In this case, we have two cases: (a) there exists a constant \( m \)-vector \( a \neq 0 \) such that \( a'g \geq 0 \) for all \( g \in \mathcal{G}(\theta_0, P_0) \) and \( P_0 \{ g \in \mathcal{G}(\theta_0, P_0) : a'g > 0 \} > 0 \), or (b) there exists \( a \neq 0 \) such that \( a'g = 0 \) for all \( g \in \mathcal{G}(\theta_0, P_0) \). For the case (a), we have \( a'E_{P_0}[g(x,\theta_0)] > 0 \), which contradicts with \( E_{P_0}[g(x,\theta_0)] = 0 \). For the case (b), we have \( a'E_{P_0}[g(x,\theta_0)g(x,\theta_0)']a = 0 \), which contradicts with Assumption 3.1 (vi).

In the second step, we show that for each \( r > 0 \), there exists \( \delta > 0 \) such that \( \theta - \theta_0 \leq \delta \) and all \( Q \in B_H(P_0, \delta) \). Pick any \( r > 0 \). From the first step, we can find \( m + 1 \) points \( \{\tilde{g}_1, \ldots, \tilde{g}_{m+1}\} = \{g(\tilde{x}_1, \theta_0), \ldots, g(\tilde{x}_{m+1}, \theta_0)\} \) in the support of \( g(x,\theta_0) \) under \( x \sim P_0 \) such that 0 is interior of the convex hull of \( \{\tilde{g}_1, \ldots, \tilde{g}_{m+1}\} \). From the property of the convex hull (Rockafeller, 1970, Corollary 2.3.1), we can take \( c_r > 0 \) such that for any points \( \{g_1, \ldots, g_{m+1}\} \) satisfying \( |g_j - \tilde{g}_j| \leq c_r \).
for \( j = 1, \ldots, m + 1 \), the interior of the convex hull of \( \{g_1, \ldots, g_{m+1}\} \) contains 0. Let us take any \( j = 1, \ldots, m + 1 \). For the second step, it is sufficient to show that there exists \( \delta_j > 0 \) such that \( Q \{g(x, \theta) - \tilde{g}_j| \leq c_r \} > 0 \) for all \( |\theta - \theta_0| \leq \delta_j \) and all \( Q \in B_H (P_0, \delta_j) \). Suppose this is false, i.e., for any \( \delta_j > 0 \), we can take a pair \((Q_j, \theta_j)\) such that \( H(Q_j, P_0) \leq \delta_j \), \(|\theta_j - \theta_0| \leq \delta_j \), and \( Q_j \{g(x, \theta_j) - \tilde{g}_j| \leq c_r \} = 0 \). Then we have

\[
\delta_j \geq H(Q_j, P_0) \geq \sqrt{\int_{\{g(x, \theta_j) - \tilde{g}_j| \leq c_r \}} (\sqrt{dQ_j - \sqrt{dP_0}})^2} = \sqrt{P_0 \{g(x, \theta_j) - \tilde{g}_j| \leq c_r \}}.
\]

On the other hand, by Assumption 3.1 (iv), the dominated convergence theorem guarantees

\[
P_0 \{g(x, \theta_j) - \tilde{g}_j| \leq c_r \} \to P_0 \{g(x, \theta_0) - \tilde{g}_j| \leq c_r \} > 0 \quad \text{as} \ \theta_j \to \theta_0.
\]

Since \( \delta_j \) can be arbitrarily small, we have a contradiction. This completes the second step.

In the third step, we show that for each \( r > 0 \), there exists \( \delta > 0 \) such that \( R_n(\theta, Q) = \inf_{P \in P \in B(Q)} H(P, Q) \) has a minimum on \( \{\theta \in \Theta : |\theta - \theta_0| \leq \delta\} \) for all \( Q \in B_H \left(P_0, \frac{\delta}{\sqrt{n}}\right) \) and all \( n \) large enough. Let us take \( \delta > 0 \) to satisfy the conclusion of the second step. By Assumption 3.1 (iv), we can take \( N_\delta \) to satisfy \( \max_{1 \leq j \leq m+1} \sup_{\theta \in \Theta \{n}} |\theta - \theta_0| \leq \delta \), \( n \in N_\delta \). Thus, letting \( G_n(\theta, Q) \) be the convex hull of the support of \( g_n(x, \theta) \) under \( x \sim Q \), the second step also guarantees that for each \( r > 0 \), there exists \( \delta > 0 \) such that \( 0 \in \text{int}\mathcal{G}_n(\theta, Q) \) for all \( |\theta - \theta_0| \leq \delta \), \( Q \in B_H (P_0, \delta) \), and all \( n \geq N_\delta \). Based on this, the convex duality result in Borwein and Lewis (1993, Theorem 3.4) implies \( R_n(\theta, Q) = \sup_{\gamma \in \mathbb{R}^m} -\int (1 + \gamma g_n(x, \theta)) dQ \) for all \( |\theta - \theta_0| \leq \delta \), \( Q \in B_H (P_0, \delta) \), and all \( n \geq N_\delta \). Since \( \sup_{\gamma \in \mathbb{R}^m} -\int (1 + \gamma g_n(x, \theta)) dQ \) is continuous at all \( \theta \) with \( |\theta - \theta_0| \leq \delta \) (by the maximum theorem), the Weierstrass theorem completes the third step.

Finally, based on the third step, it is sufficient for the conclusion to show that for every \( r > 0 \), there exists \( N \in \mathbb{N} \) such that \( R_n(\theta_0, Q) < \inf_{\theta \in \Theta \{n}} R_n(\theta, Q) \) for all \( n \geq N \) and all \( Q \in B_H \left(P_0, \frac{\delta}{\sqrt{n}}\right) \).

Pick any \( r > 0 \). We first derive an upper bound of \( R_n(\theta, Q) = \sup_{\gamma \in \mathbb{R}^m} -\int (1 + \gamma g_n(x, \theta)) dQ \). From Lemma A.5(ii), \( \gamma_n(\theta, Q) = \arg \max_{\gamma \in \mathbb{R}^m} -\int (1 + \gamma g_n(x, \theta)) dQ \) exists and \( \sup_{x \in X} |\gamma_n(\theta, Q)' g_n(x, \theta)| \leq \frac{1}{2} \) for all \( n \) large enough and all \( Q \in B_H \left(P_0, \frac{\delta}{\sqrt{n}}\right) \). Thus, by a second-order expansion around \( \gamma_n(\theta, Q) = 0 \), we have

\[
R_n(\theta, Q) \leq -1 + \gamma_n(\theta, Q)' \int g_n(x, \theta) dQ.
\]

Define \( C^* = \inf_{\theta \in \Theta \{n} |E_{P_0}[g(x, \theta)]|^2 / (1 + |E_{P_0}[g(x, \theta)]|) > 0 \). From Lemma A.3 and \( m_n n^{-1/2} \to 0 \), it holds

\[
m_n (R(\theta_0, Q) + 1) \leq m_n \left|\gamma_n(\theta_0, Q)' \int g_n(x, \theta_0) dQ\right| < C^* / 4,
\]

(A.1)
for all \( n \) large enough and all \( Q \in B_H \left( P_0, \frac{r}{\sqrt{n}} \right) \). We now derive a lower bound of \( R_n (\theta, Q) \) with \(|\theta - \theta_0| > \delta\). Pick any \( \theta \in \Theta \) such that \(|\theta - \theta_0| > \delta\), and take any \( n \) large enough and \( Q \in B_H \left( P_0, \frac{r}{\sqrt{n}} \right) \) to satisfy (A.1). If \( 0 \notin G_n (\theta, Q) \), then \( R_n (\theta, Q) = +\infty \). Thus, we concentrate on the case of \( 0 \in G_n (\theta, Q) \), which guarantees \( R_n (\theta, Q) = \sup_{\gamma \in \mathbb{R}^m} - \int \frac{1}{(1 + \gamma g_n (x, \theta))} dQ \) (Borwein and Lewis, 1993, Theorem 3.4). Let \( \gamma_0 (\theta) = E_{P_0} [g (x, \theta)] / (1 + |E_{P_0} [g (x, \theta)]|) \). Observe that

\[
R_n (\theta, Q) \geq - \int \frac{1}{(1 + m_n^{-1} \gamma_0 (\theta) g_n (x, \theta))} dQ
\]

\[
= -1 + m_n^{-1} \gamma_0 (\theta) \left( \int g_n (x, \theta) dQ - m_n^{-2} \int \frac{(\gamma_0 (\theta) g_n (x, \theta))^2}{(1 + t(x) m_n^{-1} \gamma_0 (\theta) g_n (x, \theta))} dQ \right),
\]

where the second equality follows from an expansion \((t(x) \in (0, 1) \text{ for almost every } x \text{ under } Q)\). From a similar argument to Lemma A.5 with \( \sup_{\theta \in \Theta} |\gamma_0 (\theta)| \leq 1 \) and \( m_n \to \infty \),

\[
\sup_{\theta \in \Theta} \left| \int g_n (x, \theta) dQ - \int g (x, \theta) dP_0 \right| \leq C^*, \quad m_n \sup_{\theta \in \Theta} \left| \int \frac{(\gamma_0 (\theta) g_n (x, \theta))^2}{(1 + t_1(x) m_n^{-1} \gamma_0 (\theta) g_n (x, \theta))} dQ \right| \leq C^* \frac{1}{4},
\]

for all \( n \) large enough and all \( Q \in B_H \left( P_0, \frac{r}{\sqrt{n}} \right) \). Combining these results and using the definition of \( C^* \), we obtain

\[
(A.2) \quad \inf_{\theta \in \Theta : |\theta - \theta_0| > \delta} m_n (R_n (\theta, Q) + 1) \geq \frac{C^*}{2},
\]

for all \( n \) large enough and all \( Q \in B_H \left( P_0, \frac{r}{\sqrt{n}} \right) \). Therefore, (A.1) and (A.2) complete the proof of the final step.

**Proof of (ii).** Pick arbitrary \( r > 0 \) and sequence \( Q_n \in B_H \left( P_0, r/\sqrt{n} \right) \). From the triangle inequality,

\[
(A.3) \quad \sup_{\theta \in \Theta} |E_{Q_n} [g_n (x, \theta)] - E_{P_0} [g (x, \theta)]| \leq \sup_{\theta \in \Theta} |E_{Q_n} [g_n (x, \theta)] - E_{P_0} [g_n (x, \theta)]| + \sup_{\theta \in \Theta} |E_{P_0} [g (x, \theta) I \{ x \notin X_n \}]|.
\]

The first term of (A.3) satisfies

\[
\sup_{\theta \in \Theta} |E_{Q_n} [g_n (x, \theta)] - E_{P_0} [g_n (x, \theta)]| \leq \sup_{\theta \in \Theta} \int g_n (x, \theta) \left( dQ_n^{1/2} - dP_0^{1/2} \right)^2 + 2 \sup_{\theta \in \Theta} \left| \int g_n (x, \theta) dP_0^{1/2} \left( dQ_n^{1/2} - dP_0^{1/2} \right) \right| \leq m_n r^2 \frac{r}{n} + 2 \sqrt{E_{P_0} \left( \sup_{\theta \in \Theta} |g (x, \theta)|^2 \right) r \sqrt{n}} = O \left( n^{-1/2} \right),
\]

where the first inequality follows from the triangle inequality, the second inequality follows from \( Q_n \in B_H \left( P_0, r/\sqrt{n} \right) \) and the Cauchy-Schwarz inequality, and the equality follows from Assumption
The second term of (A.3) satisfies
\[
\sup_{\theta \in \Theta} |E_{P_0} [g(x, \theta) I \{x \notin \mathcal{X}_n\}]|
\leq \left( \int \sup_{\theta \in \Theta} |g(x, \theta)|^n \, dP_0 \right)^{1/\eta} \left( \int I \{x \notin \mathcal{X}_n\} \, dP_0 \right)^{(n-1)/\eta}
\]
where the first inequality follows from the Hölder inequality, and the second inequality follows from the Markov inequality, and the equality follows from Assumption 3.1 (v) and (vii). Combining these results, we obtain the uniform convergence \(\sup_{\theta \in \Theta} |E_{Q_n} [g_n(x, \theta)] - E_{P_0} [g(x, \theta)]| \to 0\). Therefore, from the triangle inequality and \(|E_{Q_n} [g_n(x, \hat{T}_{Q_n})]| = O(n^{-1/2})\) (Lemma A.6 (i)),
\[
|E_{P_0} [g(x, \hat{T}_{Q_n})]| \leq |E_{P_0} [g(x, \hat{T}_{Q_n})] - E_{Q_n} [g_n(x, \hat{T}_{Q_n})]| + |E_{Q_n} [g_n(x, \hat{T}_{Q_n})]| \to 0.
\]
The conclusion follows from Assumption 3.1 (iii).

**Lemma A.2.** Suppose that Assumption 3.1 holds. Then for each \(r > 0\) and sequence \(Q_n \in B_H (P_0, r/\sqrt{n})\),
\[
\sqrt{n} (\hat{T}_{Q_n} - \theta_0) = -\sqrt{n} \Sigma^{-1} \int \Lambda_n dQ_n + o(1).
\]

**Proof.** The proof is based on Rieder (1994, proofs of Theorems 6.3.4 and Theorem 6.4.5). Pick arbitrary \(r > 0\) and \(Q_n \in B_H (P_0, r/\sqrt{n})\). Observe that
\[
\left\|d Q_{1/2}^{1/2} - d \hat{P}_{\theta_0, Q_n}^{1/2} + \frac{1}{2} (\hat{T}_{Q_n} - \theta_0)' \Lambda_n d Q_{1/2}^{1/2} \right\|^2
\]
\[
\begin{align*}
&= \left\|d Q_{1/2}^{1/2} - d \hat{P}_{\theta_0, Q_n}^{1/2} + \frac{1}{2} \psi_n' \Lambda_n d Q_{1/2}^{1/2} \right\|^2 + \left\| \frac{1}{2} (\hat{T}_{Q_n} - \theta_0 - \psi_n, Q_n)' \Lambda_n d Q_{1/2}^{1/2} \right\|^2 \\
&\quad + \left\{ \int \left( d Q_{1/2}^{1/2} - d \hat{P}_{\theta_0, Q_n}^{1/2} + \frac{1}{2} \psi_n' \Lambda_n d Q_{1/2}^{1/2} \right) \Lambda_n' d Q_{1/2}^{1/2} \right\} (\hat{T}_{Q_n} - \theta_0 - \psi_n, Q_n)
\end{align*}
\]
where the second equality follows from
\[
\int \left\{ d Q_{1/2}^{1/2} - d \hat{P}_{\theta_0, Q_n}^{1/2} + \frac{1}{2} \psi_n' \Lambda_n d Q_{1/2}^{1/2} \right\} \Lambda_n' d Q_{1/2}^{1/2}
\]
\[
= \int \Lambda_n' \left\{ d Q_{1/2}^{1/2} - d \hat{P}_{\theta_0, Q_n}^{1/2} \right\} d Q_{1/2}^{1/2} + \frac{1}{2} \psi_n' \Lambda_n \int \Lambda_n \Lambda_n' d Q_n = 0.
\]
The left hand side of (A.6) satisfies

$$\left\| dQ_n^{1/2} - dP_{\theta_0,Q_n}^{1/2} + \frac{1}{2} (\bar{T}_{Q_n} - \theta_0)' \Lambda_n dQ_n^{1/2} \right\| \leq \left\| dQ_n^{1/2} - dP_{\theta_0,Q_n}^{1/2} \right\| + o \left( |\bar{T}_{Q_n} - \theta_0| \right) + o \left( n^{-1/2} \right)$$

$$\leq \left\| dQ_n^{1/2} - dP_{\theta_0,Q_n}^{1/2} \right\| + o \left( |\bar{T}_{Q_n} - \theta_0| \right) + o \left( n^{-1/2} \right)$$

(A.7) $$\leq \left\| dQ_n^{1/2} - dP_{\theta_0,Q_n}^{1/2} + \frac{1}{2} \psi'_n,Q_n \Lambda_n dQ_n^{1/2} \right\| + o \left( |\bar{T}_{Q_n} - \theta_0| \right) + o \left( |\psi_n,Q_n| \right) + o \left( n^{-1/2} \right),$$

where the first inequality follows from the triangle inequality and Lemma A.3 (i), the second inequality follows from $T_{Q_n} = \arg \min_{\theta \in \Theta} \left\| dQ_n^{1/2} - dP_{\theta,Q_n}^{1/2} \right\|$, and the third inequality follows from the triangle inequality and Lemma A.3 (ii). From (A.6) and (A.7),

$$\left\| dQ_n^{1/2} - dP_{\theta_0,Q_n}^{1/2} + \frac{1}{2} \psi'_n,Q_n \Lambda_n dQ_n^{1/2} \right\|^2 + \left\| \frac{1}{2} (\bar{T}_{Q_n} - \theta_0 - \psi_n,Q_n)' \Lambda_n dQ_n^{1/2} \right\|^2 \leq \left\| dQ_n^{1/2} - dP_{\theta_0,Q_n}^{1/2} \right\|^2 + o \left( |\bar{T}_{Q_n} - \theta_0| \right) + o \left( |\psi_n,Q_n| \right) + o \left( n^{-1/2} \right).$$

This implies

$$o \left( |\bar{T}_{Q_n} - \theta_0| \right) + o \left( |\psi_n,Q_n| \right) + o \left( n^{-1/2} \right) \geq \sqrt{\frac{1}{4} (T_{Q_n} - \theta_0 - \psi_n,Q_n)' \int \Lambda_n \Lambda_n' dQ_n (T_{Q_n} - \theta_0 - \psi_n,Q_n) \geq C |\bar{T}_{Q_n} - \theta_0 - \psi_n,Q_n|,}$$

for all $n$ large enough, where the second inequality follows from Lemma A.3 (i) and Assumption 3.1 (vi).

We now analyze $\psi_n,Q_n$. From the definition of $\psi_n,Q_n$,

$$\psi_n,Q_n = -2 \left\{ \left( \int \Lambda_n \Lambda_n' dQ_n \right)^{-1} - \Sigma^{-1} \right\} \int \Lambda_n \left\{ dQ_n^{1/2} - dP_{\theta_0,Q_n}^{1/2} \right\} dQ_n^{1/2}$$

(A.9) $$= -2 \Sigma^{-1} \int \Lambda_n \left\{ dQ_n^{1/2} - dP_{\theta_0,Q_n}^{1/2} \right\} dQ_n^{1/2}.$$
From this and Lemma $\text{A.5\; (i)}$, the first term of (A.9) is $o\left(n^{-1/2}\right)$. The second term of (A.9) satisfies

$$-2\Sigma^{-1} \int \Lambda_n \left\{ dQ_n^{1/2} - dP_{\theta_0,Q_n}^{1/2} \right\} dQ_n^{1/2}$$

$$= -2\Sigma^{-1} G' \Omega^{-1} \left( \int g_n(x, \theta_0) g_n(x, \theta_0)' dQ_n \right) \gamma_n(\theta_0, Q_n)$$

$$+ 2\Sigma^{-1} G' \Omega^{-1} \left( \int \frac{\gamma_n(\theta_0, Q_n)' g_n(x, \theta_0)}{1 + \gamma_n(\theta_0, Q_n)' g_n(x, \theta_0)} g_n(x, \theta_0) g_n(x, \theta_0)' dQ_n \right) \gamma_n(\theta_0, Q_n)$$

$$= -\Sigma^{-1} G' \Omega^{-1} \left\{ \int g_n(x, \theta_0) dQ_n + \frac{1}{2} \int g_n(x, \theta_0, Q_n) g_n(x, \theta_0) dQ_n \right\} + o\left(n^{-1/2}\right)$$

$$= -\Sigma^{-1} \int \Lambda_n dQ_n + o\left(n^{-1/2}\right),$$

where the first equality follows from (A.10), the second equality follows from (A.11) and Lemma $\text{A.5\; (i)}$, and the third equality follows from Lemma $\text{A.5\; (i)}$. Therefore,

$$\sqrt{n} \psi_{n,Q_n} = -\sqrt{n} \Sigma^{-1} \int \Lambda_n dQ_n + o\left(1\right),$$

which also implies $|\psi_{n,Q_n}| = O\left(n^{-1/2}\right)$ (by Lemma $\text{A.5\; (i)}$). Combining this with (A.8),

$$\sqrt{n} \left( \hat{T}_{Q_n} - \theta_0 \right) = \sqrt{n} \psi_{n,Q_n} + o\left(\sqrt{n} |\hat{T}_{Q_n} - \theta_0| \right) + o\left(1\right).$$

By solving this equation for $\sqrt{n} \left( \hat{T}_{Q_n} - \theta_0 \right)$, the conclusion is obtained.

**Lemma A.3.** Suppose that Assumption $\text{A.1}$ holds. Then for each $r > 0$ and sequence $Q_n \in B_H\left(P_0, r/\sqrt{n}\right)$,

(i): $\left\| d\hat{P}_{\theta_0, Q_n}^{1/2} - d\hat{P}_{\theta_0, Q_n}^{1/2} + \frac{1}{2} \left( \hat{T}_{Q_n} - \theta_0 \right)' \Lambda_n dQ_n^{1/2} \right\| = o\left( \left| \hat{T}_{Q_n} - \theta_0 \right| \right) + o\left(n^{-1/2}\right),$ 

(ii): $\left\| d\hat{P}_{\theta_0, \psi_{n,Q_n}, Q_n}^{1/2} - d\hat{P}_{\theta_0, Q_n}^{1/2} + \frac{1}{2} \psi_{n,Q_n}' \Lambda_n dQ_n^{1/2} \right\| = o\left( |\psi_{n,Q_n}| \right) + o\left(n^{-1/2}\right).$

**Proof of (i).** From the convex duality of partially finite programming (Borwein and Lewis (1993)), the Radon-Nikodym derivative $d\hat{P}_{\theta,Q}/dQ$ is written as

$$d\hat{P}_{\theta,Q}/dQ = \frac{1}{(1 + \gamma_n(\theta, Q)' g_n(x, \theta))^2},$$

for each $n \in \mathbb{N}$, $\theta \in \Theta$, and $Q \in \mathcal{M}$, where $\gamma_n(\theta, Q)$ solves

$$0 = \int \frac{g_n(x, \theta)}{(1 + \gamma_n(\theta, Q)' g_n(x, \theta))^2} dQ = E_Q \left[ g_n(x, \theta) \left\{ 1 - 2\gamma_n(\theta, Q)' g_n(x, \theta) + g_n(x, \theta, Q) \right\} \right],$$

with

$$g_n(x, \theta, Q) = \frac{3 \left( \gamma_n(\theta, Q)' g_n(x, \theta) \right)^2 + 2 \left( \gamma_n(\theta, Q)' g_n(x, \theta) \right)^3}{(1 + \gamma_n(\theta, Q)' g_n(x, \theta))^2}.\]
For proof of (ii).

Denote \( t_n = T_{Q_n} - \theta_0 \). Pick arbitrary \( r > 0 \) and sequence \( Q_n \in B_H (P_0, r/\sqrt{n}) \). From the triangle inequality and (A.10),

\[
\begin{align*}
&\left\| d\hat{P}_{1/2}^{1/2} - d\hat{P}_{\theta_0,Q_n}^{1/2} + \frac{1}{2} t_n' A_n dQ_n^{1/2} \right\| \\
\leq & \left\| \left\{ \gamma_n (\theta_0, Q_n)' g_n (x, \theta_0) - \gamma_n (\bar{T}_{Q_n}, Q_n)' g_n (x, \bar{T}_{Q_n}) \right\} dQ_n^{1/2} + \frac{1}{2} t_n' A_n dQ_n^{1/2} \right\| \\
& + \left\| \left\{ \frac{1}{1+\gamma_n (\bar{T}_{Q_n}, Q_n)' g_n (x, \bar{T}_{Q_n})} (1+\gamma_n (\theta_0, Q_n)' g_n (x, \theta_0)) - 1 \right\} dQ_n^{1/2} \right\| = T_1 + T_2.
\end{align*}
\]

For \( T_2 \), Lemmas A.5 and A.6 imply \( T_2 = o (n^{-1/2}) \). For \( T_1 \), the triangle inequality and (A.11) yield

\[
T_1 \leq \left\| \begin{array}{c}
-\frac{1}{2} E_{\bar{T}_{Q_n}} \left[ g_n (x, \bar{T}_{Q_n}) \right]' E_{\bar{T}_{Q_n}} \left[ g_n (x, \bar{T}_{Q_n}) g_n (x, \bar{T}_{Q_n}) \right]^{-1} g_n (x, \bar{T}_{Q_n}) \\
+\frac{1}{2} E_{\bar{T}_{Q_n}} \left[ g_n (x, \theta_0) \right]' E_{\bar{T}_{Q_n}} \left[ g_n (x, \theta_0) g_n (x, \theta_0) \right]^{-1} g_n (x, \theta_0) + \frac{1}{2} t_n A_n
\end{array} \right\} dQ_n^{1/2} \\
\leq \left\| \begin{array}{c}
E_{\bar{T}_{Q_n}} \left[ g_n (x, \theta_0) \right]' E_{\bar{T}_{Q_n}} \left[ g_n (x, \theta_0) g_n (x, \theta_0) \right]^{-1} \left\{ g_n (x, \bar{T}_{Q_n}) - g_n (x, \theta_0) \right\} dQ_n^{1/2} \\
+ \frac{1}{2} t_n' \left( \int \frac{\partial g_n (x, \theta)}{\partial \theta} dQ_n - G \right)' E_{\bar{T}_{Q_n}} \left[ g_n (x, \theta_0) g_n (x, \theta_0) \right]^{-1} g_n (x, \theta_0) dQ_n^{1/2} \\
+ \frac{1}{2} t_n' G' (\Omega^{-1} - E_{\bar{T}_{Q_n}} \left[ g_n (x, \theta_0) g_n (x, \theta_0) \right]^{-1}) g_n (x, \theta_0) dQ_n^{1/2}
\end{array} \right\| \\
= o \left( n^{-1/2} \right) + o (t_n),
\]

where \( \hat{\theta} \) is a point on the line joining \( \theta_0 \) and \( \bar{T}_{Q_n} \), and the equality follows from Lemmas A.5 (i) and A.6 (i).

**Proof of (ii).** Similar to the proof of Part (i) of this lemma.
Lemma A.4. Suppose that Assumption 3.1 hold. Then for each \( t \in \mathbb{R}^p \),

(i): \(|E_{P_0} [g_n (x, \theta_0)]| = o \left(n^{-1/2}\right), |E_{P_0} [g_n (x, \theta_n)]| = O \left(n^{-1/2}\right), |E_{P_0} \left[g_n (x, \theta_n) g_n (x, \theta_n)'\right] - \Omega| = o (1), \text{ and } |E_{P_0} \left[\partial g_n (x, \theta_n) / \partial \theta'\right] - G| = o (1),

(ii): \( \gamma_n (\theta_n, P_0) = \arg \max_{\gamma \in \mathbb{R}^m} - \int (1 + \gamma g_n (x, \theta_n) \, dP_0 \text{ exists for all } n \text{ large enough, } |\gamma_n (\theta_n, P_0)| = O \left(n^{-1/2}\right), \text{ and } \sup_{x \in \mathcal{X}} |\gamma_n (\theta_n, P_0)' g_n (x, \theta_n)| = o (1). \)

Proof of (i). Proof of the first statement. The same argument as (A.4) with Assumption 3.1 (iii) yields the conclusion.

Proof of the second statement. Pick an arbitrary \( t \in \mathbb{R}^p \). From the triangle inequality,

\[
|E_{P_0} [g_n (x, \theta_n)]| \leq |E_{P_0} [g (x, \theta_n) 1 \{x \notin X_n\}]| + |E_{P_0} [g (x, \theta_n)]|.
\]

By the same argument as (A.4) and \( E_{P_0} [[g (x, \theta_n)]^\gamma] < \infty \) (from Assumption 3.1 (v)), the first term of (A.12) is \( o \left(n^{-1/2}\right) \). The second term of (A.12) satisfies

\[
|E_{P_0} [g (x, \theta_n)]| \leq E_{P_0} \left[\sup_{\theta \in \mathcal{N}} \left|\frac{\partial g (x, \theta)}{\partial \theta'}\right| \right] \left|\frac{t}{\sqrt{n}}\right| = O \left(n^{-1/2}\right),
\]

for all \( n \) large enough, where the inequality follows from a Taylor expansion around \( t = 0 \) and Assumption 3.1 (iii), and the equality follows from Assumption 3.1 (v). Combining these results, the conclusion is obtained.

Proof of the third statement. Pick an arbitrary \( t \in \mathbb{R}^p \). From the triangle inequality,

\[
|E_{P_0} \left[g_n (x, \theta_n) g_n (x, \theta_n)'\right] - \Omega| \leq |E_{P_0} \left[g_n (x, \theta_n) g_n (x, \theta_n)'\right] - E_{P_0} \left[g (x, \theta_n) g (x, \theta_n)'\right]| + |E_{P_0} \left[g (x, \theta_n) g (x, \theta_n)'\right] - \Omega|.
\]

The first term is \( o \left(n^{-1/2}\right) \) by the same argument as (A.4) and the second term converges to zero by the continuity of \( g (x, \theta) \) at \( \theta_0 \).

Proof of the fourth statement. Similar to the proof of the third statement.

Proof of (ii). Pick an arbitrary \( t \in \mathbb{R}^p \). Let \( \Gamma_n = \{ \gamma \in \mathbb{R}^m : |\gamma| \leq a_n \} \) with a positive sequence \( \{a_n\}_{n \in \mathbb{N}} \) satisfying \( a_n m_n \to 0 \) and \( a_n n^{1/2} \to \infty \). Observe that

\[
\sup_{\gamma \in \Gamma_n, x \in \mathcal{X}, \theta \in \Theta} |g_n (x, \theta)| \leq a_n m_n \to 0.
\]
Since \( R_n (P_0, \theta_n, \gamma) \) is twice continuously differentiable with respect to \( \gamma \) and \( \Gamma_n \) is compact, \( \hat{\gamma} = \arg \max_{\gamma \in \Gamma_n} R_n (P_0, \theta_n, \gamma) \) exists for each \( n \in \mathbb{N} \). A Taylor expansion around \( \hat{\gamma} = 0 \) yields

\[
-1 = R_n (P_0, \theta_n, 0) \leq \frac{d^2 R_n (P_0, \theta_n, \hat{\gamma})}{d \gamma^2} \hat{\gamma}^2 = -1 + \hat{\gamma}' E_P_0 [g_n (x, \theta_n)] - \hat{\gamma}' E_P_0 \left[ \frac{g_n (x, \theta_n) g_n (x, \theta_n)}{(1 + \hat{\gamma}' g_n (x, \theta_n))^2} \right] \hat{\gamma} \\
\leq -1 + \hat{\gamma}' E_P_0 [g_n (x, \theta_n)] - C \hat{\gamma}' E_P_0 [g_n (x, \theta_n) g_n (x, \theta_n)] \hat{\gamma} \\
(A.14) \leq -1 + |\hat{\gamma}| |E_P_0 [g_n (x, \theta_n)]| - C |\hat{\gamma}|^2,
\]

for all \( n \) large enough, where \( \hat{\gamma} \) is a point on the line joining 0 and \( \hat{\gamma} \), the second inequality follows from \( \text{Lemma A.4} \) (i) and Assumption \( 3.1 \) (vi). Thus, Lemma \( \text{A.4} \) (i) implies

\[
(A.15) \quad C |\hat{\gamma}| \leq |E_P_0 [g_n (x, \theta_n)]| = O \left( n^{-1/2} \right).
\]

From \( a_n n^{1/2} \to \infty \), \( \hat{\gamma} \) is an interior point of \( \Gamma_n \) and satisfies the first-order condition \( \partial R_n (Q_n, \theta_0, \hat{\gamma}) / \partial \gamma = 0 \) for all \( n \) large enough. Since \( R_n (Q_n, \theta_0, \gamma) \) is concave in \( \gamma \) for all \( n \) large enough, \( \hat{\gamma} = \arg \max_{\gamma \in \mathbb{R}^m} R_n (P_0, \theta_n, \gamma) \) for all \( n \) large enough and the first statement is obtained. Thus, the second statement is obtained from \( (A.15) \). The third statement follows from \( (A.15) \) and Assumption \( 3.1 \) (vii).

**Lemma A.5.** Suppose that Assumption \( 3.1 \) holds. Then for each \( r > 0 \) and sequence \( Q_n \in B_H (P_0, r/\sqrt{n}) \),

(i): \( |E_{Q_n} [g_n (x, \theta_0)]| = O \left( n^{-1/2} \right) \), and \( |E_{Q_n} [g_n (x, \theta_0) g_n (x, \theta_0)']| - \Omega = o (1) \),

(ii): \( \gamma_n (\theta_0, Q_n) = \arg \max_{\gamma \in \mathbb{R}^m} - \int \frac{1}{1 + \gamma g_n (x, \theta_0)} dQ_n \) exists for all \( n \) large enough, and \( |\gamma_n (\theta_0, Q_n)| = O \left( n^{-1/2} \right) \), and \( \sup_{x \in \mathcal{X}} |\gamma_n (\theta_0, Q_n)' g_n (x, \theta_0)| = o (1) \).

**Proof of (i).** **Proof of the first statement.** Pick any \( r > 0 \) and sequence \( Q_n \in B_H (P_0, r/\sqrt{n}) \).

We have

\[
|E_{Q_n} [g_n (x, \theta_0)]| \\
\leq \left| \int g_n (x, \theta_0) \{ dQ_n - dP_0 \} \right| + |E_{P_0} [g_n (x, \theta_0)]| \\
\leq \left| \int g_n (x, \theta_0) \{ dQ_n^{1/2} - dP_0^{1/2} \}^2 \right| + 2 \left| \int g_n (x, \theta_0) dP_0^{1/2} \left\{ dQ_n^{1/2} - dP_0^{1/2} \right\} \right| + o \left( n^{-1/2} \right) \\
\leq m_n \frac{r^2}{n} + 2E_{P_0} \left[ |g (x, \theta_0)|^2 \right] \frac{r}{\sqrt{n}} + o \left( n^{-1/2} \right) = O \left( n^{-1/2} \right),
\]

where the first and second inequalities follow from the triangle inequality and Lemma \( \text{A.4} \) (i), the third inequality follows from the Cauchy-Schwarz inequality and \( Q_n \in B_H (P_0, r/\sqrt{n}) \), and the equality follows from Assumption \( 3.1 \) (v) and (vii).
Proof of the second statement. Pick arbitrary \( r > 0 \) and sequence \( Q_n \in B_H (P_0, r/\sqrt{n}) \). From the triangle inequality,

\[
(A.16) \left| E_{Q_n} \left[ g_n (x, \theta_0) g_n (x, \theta_0) \right] - \Omega \right| \\
\leq \left| E_{Q_n} \left[ g_n (x, \theta_0) g_n (x, \theta_0) \right] - E_{P_0} \left[ g_n (x, \theta_0) g_n (x, \theta_0) \right] \right| + \left| E_{P_0} \left[ g (x, \theta_0) g (x, \theta_0) \right] I \{ x \notin \chi_n \} \right|.
\]

The first term of the RHS of (A.16) satisfies

\[
\begin{align*}
&\left| E_{Q_n} \left[ g_n (x, \theta_0) g_n (x, \theta_0) \right] - E_{P_0} \left[ g_n (x, \theta_0) g_n (x, \theta_0) \right] \right| \\
&\leq \int g_n (x, \theta_0) g_n (x, \theta_0) \left\{ dQ_n^{1/2} - dP_0^{1/2} \right\}^2 + 2 \int g_n (x, \theta_0) g_n (x, \theta_0) dP_0^{1/2} \left\{ dQ_n^{1/2} - dP_0^{1/2} \right\} \\
&\leq m_n^2 r^2 + 2E_{P_0} \left[ |g (x, \theta_0)|^4 \right] \frac{r}{\sqrt{n}} = o (1),
\end{align*}
\]

where the first inequality follows from the triangle inequality, the second inequality follows from the Cauchy-Schwarz inequality and \( Q_n \in B_H (P_0, r/\sqrt{n}) \), and the equality follows from Assumption 3.1 (v) and (vii). The second term of (A.16) satisfies

\[
\begin{align*}
&\left| E_{P_0} \left[ g (x, \theta_0) g (x, \theta_0) \right] I \{ x \notin \chi_n \} \right| \\
&\leq \left( \int |g (x, \theta_0) g (x, \theta_0)|^{1+\delta} dP_0 \right)^{\frac{1}{1+\delta}} \left( \int \Omega \{ x \notin \chi_n \} dP_0 \right)^{\frac{\delta}{1+\delta}} \\
&\leq \left( E_{P_0} \left[ |g (x, \theta_0)|^{2+\delta} \right] \right)^{\frac{1}{1+\delta}} \left( m_n^{-\eta} E_{P_0} \left[ |g (x, \theta)|^\eta \right] \right)^{\frac{\delta}{1+\delta}} = o (1),
\end{align*}
\]

for sufficiently small \( \delta > 0 \), where the first inequality follows from the Hölder inequality, the second inequality follows from the Markov inequality, and the equality follows from Assumption 3.1 (vii).

Proof of (ii). Similar to the proof of Lemma A.4 (ii). Repeat the same argument with \( R_n (Q_n, \theta_0, \gamma) \) instead of \( R_n (P_0, \theta, \gamma) \).

Lemma A.6. Suppose that Assumption 3.1 holds. Then for each \( r > 0 \) and sequence \( Q_n \in B_H (P_0, r/\sqrt{n}) \),

(i): \[ E_{Q_n} \left[ g_n (x, \tilde{T}_{Q_n}) \right] = O \left( n^{-1/2} \right), \quad E_{Q_n} \left[ g_n (x, \tilde{T}_{Q_n}) g_n (x, \tilde{T}_{Q_n}) \right] - \Omega = o (1), \quad \text{and} \quad E_{Q_n} \left[ \frac{\partial g_n (x, \tilde{T}_{Q_n})}{\partial \theta} - G \right] = o (1), \]

(ii): \[ \gamma_n (\tilde{T}_{Q_n}, Q_n) = \arg \max_{\gamma \in \mathbb{R}^m} - \int \frac{1}{1 + \gamma' g_n (x, \tilde{T}_{Q_n})} dQ_n \] exists for all \( n \) large enough, \[ \gamma_n (\tilde{T}_{Q_n}, Q_n) = O \left( n^{-1/2} \right), \quad \text{and} \quad \sup_{x \in \mathcal{X}} |\gamma_n (\tilde{T}_{Q_n}, Q_n)' g_n (x, \tilde{T}_{Q_n})| = o (1). \]

Proof of (i). Proof of the first statement. Pick any \( r > 0 \) and sequence \( Q_n \in B_H (P_0, r/\sqrt{n}) \).

Define \( \tilde{\gamma} = \frac{E_{Q_n} \left[ g_n (x, \tilde{T}_{Q_n}) \right]}{\sqrt{m} E_{Q_n} \left[ g_n (x, \tilde{T}_{Q_n}) \right]} \). Since |\( \tilde{\gamma} \)| = \( n^{-1/2} \),

\[
(A.17) \sup_{x \in \mathcal{X}, \theta \in \Theta} |\tilde{\gamma}' g_n (x, \theta)| \leq n^{-1/2} m_n \to 0.
\]
Observe that

\[(A.18)\]

\[
\left| E_{Q_n} \left[ g_n (x, \tilde{T}_{Q_n}) g_n (x, \tilde{T}_{Q_n})' \right] \right| 
\leq \int \sup_{\theta \in \Theta} |g_n (x, \theta)|^2 \left\{ dQ_n^{1/2} - dP_0^{1/2} \right\}^2 + 2 \int \sup_{\theta \in \Theta} |g_n (x, \theta)|^2 dP_0^{1/2} \left\{ dQ_n^{1/2} - dP_0^{1/2} \right\} + E_{P_0} \left[ \sup_{\theta \in \Theta} |g_n (x, \theta)|^2 \right]
\]

\[
\leq m_n^2 r^2 / n + 2m_n \sqrt{E_{P_0} \left[ \sup_{\theta \in \Theta} |g (x, \theta)|^2 \right]} \frac{r}{\sqrt{n}} + E_{P_0} \left[ \sup_{\theta \in \Theta} |g (x, \theta)|^2 \right] \leq CE_{P_0} \left[ \sup_{\theta \in \Theta} |g (x, \theta)|^2 \right],
\]

for all \( n \) large enough, where the first inequality follows from the triangle inequality, the second inequality follows from the Cauchy-Schwarz inequality and \( Q_n \in B_H (P_0, r/\sqrt{n}) \), and the last inequality follows from Assumption 3.1 (v) and (vii). Thus, an expansion around \( \bar{\gamma} = 0 \) yields

\[(A.19)\]

\[
R_n \left( Q_n, \tilde{T}_{Q_n}, \bar{\gamma} \right) = -1 + \bar{\gamma}' E_{Q_n} \left[ g_n (x, \tilde{T}_{Q_n}) \right] - \bar{\gamma}' E_{Q_n} \left[ \frac{g_n (x, \tilde{T}_{Q_n}) g_n (x, \tilde{T}_{Q_n})'}{(1 + \bar{\gamma} g_n (x, \tilde{T}_{Q_n}))^2} \right] \bar{\gamma}
\]

\[
\geq -1 + n^{-1/2} \left| E_{Q_n} \left[ g_n (x, \tilde{T}_{Q_n}) \right] \right| - C \bar{\gamma} \left| E_{Q_n} \left[ g_n (x, \tilde{T}_{Q_n}) g_n (x, \tilde{T}_{Q_n})' \right] \right| \bar{\gamma}
\]

\[
\geq -1 + n^{-1/2} \left| E_{Q_n} \left[ g_n (x, \tilde{T}_{Q_n}) \right] \right| - C n^{-1},
\]

for all \( n \) large enough, where \( \bar{\gamma} \) is a point on the line joining 0 and \( \bar{\gamma} \), the first inequality follows from \( (A.17) \), and the second inequality follows from \( \bar{\gamma}' \bar{\gamma} = n^{-1} \) and \( (A.18) \). From the duality of partially finite programming (Borwein and Lewis (1993)), \( \gamma_n (\tilde{T}_{Q_n}, Q_n) \) and \( \tilde{T}_{Q_n} \) are written as \( \gamma_n (\tilde{T}_{Q_n}, Q_n) = \arg \max_{\gamma \in \mathbb{R}^m} R_n \left( Q_n, \tilde{T}_{Q_n}, \gamma \right) \) and \( \tilde{T}_{Q_n} = \arg \min_{\theta \in \Theta} R_n \left( Q_n, \theta, \gamma_n (\theta, Q_n) \right) \). Therefore, from \( (A.19) \),

\[(A.20)\]

\[
-1 + n^{-1/2} \left| E_{Q_n} \left[ g_n (x, \tilde{T}_{Q_n}) \right] \right| - C n^{-1}
\]

\[
\leq R_n \left( Q_n, \tilde{T}_{Q_n}, \bar{\gamma} \right) \leq R_n \left( Q_n, \tilde{T}_{Q_n}, \gamma_n (\tilde{T}_{Q_n}, Q_n) \right) \leq R_n \left( Q_n, \theta_0, \gamma_n (\theta_0, Q_n) \right).
\]

By a similar argument to \( (A.14) \) combined with \( |\gamma_n (\theta_0, Q_n)| = O \left( n^{1/2} \right) \) and \( |E_{Q_n} [g_n (x, \theta_0)]| = O \left( n^{1/2} \right) \) (by Lemma \( \text{A.5} \)), we have

\[(A.21)\]

\[
R_n \left( Q_n, \theta_0, \gamma_n (\theta_0, Q_n) \right) \leq -1 + |\gamma_n (\theta_0, Q_n)| |E_{Q_n} [g_n (x, \theta_0)]| - C |\gamma_n (\theta_0, Q_n)|^2 = -1 + O \left( n^{-1} \right).
\]

From \( (A.20) \) and \( (A.21) \), the conclusion follows.

**Proof of the second statement.** Similar to the proof of the second statement of Lemma \( \text{A.5} \) (i).
Proof of the third statement. Pick arbitrary $r > 0$ and sequence $Q_n \in B_H (P_0, r/\sqrt{n})$. From the triangle inequality,
\begin{equation}
|E_{Q_n} \left[ \partial g_n (x, \bar{T}_{Q_n}) / \partial \theta' \right] - G | 
\leq |E_{Q_n} \left[ \partial g_n (x, \bar{T}_{Q_n}) / \partial \theta' \right] - E_{P_0} \left[ \partial g_n (x, \bar{T}_{Q_n}) / \partial \theta' \right]|
\end{equation}
(A.22)
\begin{equation}
\quad + |E_{P_0} \left[ \mathbb{I} \{ x \notin \mathcal{X}_n \} \partial g (x, \bar{T}_{Q_n}) / \partial \theta' \right] | + |E_{P_0} \left[ \partial g (x, \bar{T}_{Q_n}) / \partial \theta' \right] - G | .
\end{equation}

The first term of (A.22) satisfies
\begin{align*}
|E_{Q_n} \left[ \partial g_n (x, \bar{T}_{Q_n}) / \partial \theta' \right] - E_{P_0} \left[ \partial g_n (x, \bar{T}_{Q_n}) / \partial \theta' \right] | & \leq \left| \int \partial g_n (x, \bar{T}_{Q_n}) / \partial \theta' \left\{ dQ_n^{1/2} - dP_0^{1/2} \right\} \right| \\
& \leq \sup_{x \in \mathcal{X}_n, \theta \in \mathcal{N}} \left| \partial g_n (x, \theta) / \partial \theta' \right| \frac{P_0}{n} + 2E_{P_0} \left[ \sup_{\theta \in \mathcal{N}} \left| \partial g_n (x, \theta) / \partial \theta' \right|^2 \right] \frac{r}{\sqrt{n}} = o (1),
\end{align*}
where the first inequality follows from the triangle inequality, the second inequality follows from the Cauchy-Schwarz inequality, and the equality follows from Assumption 3.1 (v) and (vii). The second term of (A.22) is $o (1)$ by the same argument as (A.4). The third term of (A.22) is $o (1)$ by the continuity of $\partial g (x, \theta) / \partial \theta'$ at $\theta_0$ and Lemma A.4 (ii). Therefore, the conclusion is obtained.

Proof of (ii). Similar to the proof of Lemma A.4 (ii). Repeat the same argument with $R_n (Q_n, \bar{T}_{Q_n}, \gamma)$ instead of $R_n (P_0, \theta_n, \gamma)$.

Lemma A.7. Suppose that Assumption 3.1 holds. Then for each sequence $Q_n \in B_H (P_0, r/\sqrt{n})$ and $r > 0$, $\bar{T}_{P_n} \stackrel{p}{\rightarrow} \theta_0$ under $Q_n$.

Proof. Similar to the proof of Lemma A.1 (i).

Lemma A.8. Suppose that Assumption 3.1 holds. Then for each $r > 0$ and sequence $Q_n \in B_H (P_0, r/\sqrt{n}),$
\begin{align*}
\sqrt{n} (\bar{T}_{P_n} - \theta_0) & = -\sqrt{n} \Sigma^{-1} \int \Lambda_n dP_n + o_p (1) \quad \text{under } Q_n, \\
\sqrt{n} (\bar{T}_{P_n} - \bar{T}_{Q_n}) & \overset{d}{\rightarrow} N (0, \Sigma^{-1}) \quad \text{under } Q_n.
\end{align*}

Proof. The proof of the first statement is similar to that of Lemma A.2 (replace $Q_n$ with $P_n$ and use Lemmas A.9 and A.10 instead of Lemmas A.5 and A.6). For the second statement, Lemma A.2 and the first statement imply
\begin{equation}
\sqrt{n} (\bar{T}_{P_n} - \bar{T}_{Q_n}) = -\Sigma^{-1} G' \Omega^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ g_n (x_i, \theta_0) - E_{Q_n} [g_n (x, \theta_0)] \right\} + o_p (1),
\end{equation}
Lemma A.10. Suppose that Assumption A.4 holds. Then for each $r > 0$ and sequence $Q_n \in B_H (P_0, r/\sqrt{n})$, the followings hold under $Q_n$:

(i): $|E_{P_n} [g_n (x, \theta_0)]| = O_p \left( n^{-1/2} \right)$, $|E_{P_n} [g_n (x, \theta_0) g_n (x, \theta_0)'] - \Omega | = o_p (1)$.

(ii): $\gamma_n (\theta_0, P_n) = \arg \max_{\gamma \in \mathbb{R}^m} - \frac{1}{(1+\gamma g_n(x, \theta_0))} dP_n$ exists a.s. for all $n$ large enough, $|\gamma_n (\theta_0, P_n)| = O_p \left( n^{-1/2} \right)$, and $\sup_{x \in \mathcal{X}} |\gamma_n (\theta_0, P_n) g_n (x, \theta_0) | = o_p (1)$.

**Proof of (i). Proof of the first statement.** From the triangle inequality,

$$|E_{P_n} [g_n (x, \theta_0)]| \leq |E_{P_n} [g_n (x, \theta_0)] - E_{Q_n} [g_n (x, \theta_0)]| + |E_{Q_n} [g_n (x, \theta_0)]| .$$

The first term is $O_p \left( n^{-1/2} \right)$ by the central limit theorem for the triangular array $\{g_n (x_i, \theta_0)\}_{1 \leq i \leq n, n}$. The second term is $O \left( n^{-1/2} \right)$ by Lemma A.9 (i).

**Proof of the second statement.** From the triangle inequality,

$$|E_{P_n} [g_n (x, \theta_0) g_n (x, \theta_0)'] - \Omega | \leq |E_{P_n} [g_n (x, \theta_0) g_n (x, \theta_0)'] - E_{Q_n} [g_n (x, \theta_0) g_n (x, \theta_0)']| + |E_{Q_n} [g_n (x, \theta_0) g_n (x, \theta_0)'] - \Omega | .$$

From a law of large numbers, the first term is $o_p (1)$. From Lemma A.4 (i), the second term is $o (1)$.

**Proof of (ii).** Similar to the proof of Lemma A.4 (ii) except using Lemma A.9 (i) instead of Lemma A.4 (i).

**Lemma A.10.** Suppose that Assumption A.4 holds. Then for each $r > 0$ and sequence $Q_n \in B_H (P_0, r/\sqrt{n})$, the followings hold under $Q_n$:

(i): $|E_{P_n} [g_n (x, \tilde{T}_n)]| = O_p \left( n^{-1/2} \right)$, $|E_{P_n} [g_n (x, \tilde{T}_n) g_n (x, \tilde{T}_n)'] - \Omega | = O_p \left( n^{-1/2} \right)$, and

$$|E_{P_n} [\partial g_n (x, \tilde{T}_n) / \partial \theta'] - G | = o_p (1).$$
(ii): \( \gamma_n (\tilde{T}_{P_n}, P_n) = \arg \max_{\gamma \in \mathbb{R}^m} - \int \frac{1}{1 + \gamma \cdot g_n(x, \tilde{T}_{P_n})} dP_n \) exists a.s. for all \( n \) large enough, \( \left| \gamma_n (\tilde{T}_{P_n}, P_n) \right| = O_p \left( n^{-1/2} \right) \), and \( \sup_{x \in X} \left| \gamma_n (\tilde{T}_{P_n}, P_n)' \cdot g_n(x, \tilde{T}_{P_n}) \right| = o_p(1) \).

**Proof of (i).** Similar to the proof of Lemma A.6 (i).

**Proof of (ii).** Similar to the proof of Lemma A.6 (ii).
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